Singular behaviour of certain infinite products of random 2 × 2 matrices

1983 J. Phys. A: Math. Gen. 16 2641

(http://iopscience.iop.org/0305-4470/16/12/013)

J. Phys. A: Math. Gen. 16 (1983) 2641–2654. Printed in Great Britain

Singular behaviour of certain infinite products of random 2×2 matrices

B Derrida and H J Hilhorst†

SPT, CEN Saclay, 91191 Gif-sur-Yvette Cedex, France

Received 8 February 1983

Abstract. We consider an infinite product of random matrices which appears in several physical problems, in particular the Ising chain in a random field. The random matrices depend analytically on a parameter ε in such a way that for $\varepsilon = 0$ they all commute. Under certain conditions we find that the Lyapunov index of this product behaves approximately as $C\varepsilon^{2\alpha}$ for $\varepsilon \to 0$.

Our approach is based on the decomposition of an exact integral equation for this problem into two reduced equations. We give an expression for the exponent α in terms of the probability distribution of the matrices, and for the proportionality constant C in terms of the solutions of the reduced integral equations.

In cases where exact results are available, agreement is obtained. The physical consequences for disordered one-dimensional systems are pointed out.

1. Introduction

Infinite products of random matrices appear in physics, in particular, when one studies one-dimensional disordered systems. Although, in general, very little can be said about the behaviour of such products as a function of the parameters in the problem, there are special cases where a better insight can be gained. One such case is when a parameter ε is available such that for $\varepsilon = 0$ all random matrices commute. The purpose of this work is to investigate the particular expression

$$F(\varepsilon) = \lim_{N \to \infty} \frac{1}{N} \log \operatorname{Tr} \prod_{i=1}^{N} \begin{pmatrix} 1 & \varepsilon \\ z_i\varepsilon & z_i \end{pmatrix} \tag{1}$$

where the z_i are independent positive random numbers with a given probability distribution $\rho(z)$. The quantity $F(\varepsilon)$ is often called the characteristic index or the Lyapunov exponent of the random product. The product in (1) appears in several contexts, as indicated at the end of this introduction.

The main result that we shall obtain is that the function $F(\varepsilon)$ exhibits the singular behaviour

$$F(\varepsilon) \simeq C\varepsilon^{2\alpha} \qquad \varepsilon \to 0 \tag{2}$$

provided $\rho(z)$ is such that

$$\langle \log z \rangle < 0 \qquad \langle z \rangle > 1 \tag{3a, b}$$

† On leave from the Laboratory of Applied Physics, TH Delft, Postbus 5046, 2600 GA Delft, The Netherlands.

where we use the notation

$$\langle A(z)\rangle = \int \mathrm{d}z\, \rho(z) A(z). \tag{4}$$

The exponent α is given by the positive root of

$$\langle z^{\alpha}\rangle = 1 \tag{5}$$

and will, in general, vary continuously with any of the parameters implicit in the function $\rho(z)$.

Equation (5) also occurs in the mathematical literature on random matrix products (Kesten *et al* 1975), and analyticity properties of Lyapunov indices have been proved in certain cases (Ruelle 1979). However, a singular behaviour of the type (2) has, to our knowledge, not been found before. In this work we shall discuss this behaviour, explain why α is given by (5), and show how to determine the proportionality constant C.

We remark that $\rho(z)$ is arbitrary throughout: we do not, in particular, assume any narrowness of the distribution. Our approach is based on the decomposition of the exact integral equation for this problem into two reduced equations that are sufficiently manageable. We do not justify our procedure rigorously, but expect it to be valid, for $\varepsilon \to 0$, for a wide class of distributions $\rho(z)$. In special cases where exact results can be obtained, these agree fully with ours.

To see how the product in expression (1) can occur, consider an Ising chain with nearest-neighbour interactions in a random field. The Hamiltonian of this system is

$$\mathscr{H} = -J\sigma_1\sigma_N - \sum_{i=1}^{N-1} J\sigma_i\sigma_{i+1} - \sum_{i=1}^{N} h_i\sigma_i \tag{6}$$

where the fields h_i are randomly distributed according to a given distribution $\tilde{\rho}(h)$. The partition function Z of this system can be written as

$$Z = \mathrm{Tr}\left\{\prod_{i=1}^{N}\left[\begin{pmatrix} e^{h_i} & 0 \\ 0 & e^{-h_i}\end{pmatrix}\begin{pmatrix} e^{J} & e^{-J} \\ e^{-J} & e^{J}\end{pmatrix}\right]\right\} \tag{7}$$

and can be transformed easily to give

$$Z = \left(\prod_{i=1}^{N} e^{h_i+J}\right) \mathrm{Tr}\left[\prod_{i=1}^{N}\begin{pmatrix} 1 & e^{-2J} \\ e^{-2h_i-2J} & e^{-2h_i}\end{pmatrix}\right]. \tag{8}$$

One sees that with the obvious change of variables

$$e^{-2h_i} = z_i \qquad \text{and} \qquad e^{-2J} = \varepsilon \tag{9}$$

one finds the product defined in (1). The limit $\varepsilon \to 0$ corresponds to the case where the ferromagnetic interactions become very strong compared with the temperature (here $T = 1$) and with the fluctuating field h_i.

The product (1) also occurs in the solution of a two-dimensional square Ising model with row-wise random vertical interactions, as given by McCoy (1969, 1972) and McCoy and Wu (1969, 1973). The role of the variable ε is played there by a wavenumber θ, which enters because of invariance under horizontal translations.

Lastly, the kind of behaviour described in this paper can also be found in random hopping problems (Kesten *et al* 1975, Alexander *et al* 1981, Derrida and Pomeau 1982, Bernasconi and Schneider 1982, Ziman 1982, Derrida 1983), which leads us to suppose that it should be possible to solve these in a way similar to (1).

2. A useful formula for $F(\varepsilon)$

In this section we reduce the calculation of $F(\varepsilon)$ defined by (1) to the problem of solving an integral equation. This kind of transformation is often used in the calculation of products of random matrices (see, for example, Dyson 1953, McCoy 1972, Alexander *et al* 1981).

Because all the matrices in expression (1) have positive elements, one can also calculate $F(\varepsilon)$ by

$$F(\varepsilon) = \lim_{N\to\infty} (1/N) \log(\|V_N\|/\|V_0\|) \tag{10}$$

where the vector V_0 is an arbitrary vector with positive components

$$V_0 = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \tag{11}$$

and the vectors V_i $(1 \leq i \leq N)$ and their two components a_i and b_i are defined by

$$V_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon \\ z_i\varepsilon & z_i \end{pmatrix} V_{i-1}. \tag{12}$$

The next step to transform the problem is to consider that $F(\varepsilon)$ is also given by

$$F(\varepsilon) = \lim_{N\to\infty} (1/N) \log(a_N/a_0). \tag{13}$$

This is due to the fact that the ratio b_i/a_i has an upper bound z_i/ε and a lower bound $z_i\varepsilon$ which do not increase with N. Equation (12) tells us that the a_i and the b_i obey the recursion relations

$$\begin{aligned} a_{i+1} &= a_i + \varepsilon b_i \\ b_{i+1} &= z_i\varepsilon a_i + z_i b_i. \end{aligned} \tag{14}$$

One can eliminate the b_i to find

$$a_{i+2} = (1+z_i)a_{i+1} + z_i(\varepsilon^2 - 1)a_i. \tag{15}$$

If we introduce the ratios R_i,

$$R_i = a_{i+1}/a_i, \tag{16}$$

one finds that they obey the recursion

$$R_{i+1} = 1 + z_i + z_i(\varepsilon^2 - 1)/R_i \tag{17}$$

and $F(\varepsilon)$ can be calculated from

$$F(\varepsilon) = \lim_{N\to\infty} (1/N) \sum_{i=0}^{N-1} \log R_i. \tag{18}$$

The way usually followed to calculate $F(\varepsilon)$ is to consider that when i increases, a stationary probability distribution $P(R)$ of the R_i independent of i exists (Furstenberg 1963). Therefore, one obtains a useful formula to calculate $F(\varepsilon)$:

$$F(\varepsilon)=\int P(R)\log(R)\,\mathrm{d}R \tag{19}$$

where the stationary distribution $P(R)$ is the solution of the integral equation (see equation (17))

$$\begin{aligned}P(R)&=\int \rho(z)\,\mathrm{d}z\int P(R')\,\mathrm{d}R'\,\delta(R-1-z-z(\varepsilon^2-1)/R')\\&=\int \rho(z)\,\mathrm{d}z\frac{(1-\varepsilon^2)z}{(z+1-R)^2}P\left(\frac{(1-\varepsilon^2)z}{z+1-R}\right).\end{aligned} \tag{20}$$

All that has been said in this section can be generalised to other products of random matrices with positive elements. However, one usually does not know how to calculate analytically either the logarithm of the trace of the product ($F(\varepsilon)$) or the stationary distribution ($P(R)$). What makes the problem simpler is that we will limit ourselves to the case $\varepsilon \ll 1$, i.e. to the neighbourhood of $\varepsilon=0$, at which point the matrices in expression (1) commute. For $\varepsilon=0$ one easily finds that

$$\left(\prod_{i=1}^{N} z_i\right)^{1/N}=\exp\left(N^{-1}\sum_{i=1}^{N}\log z_i\right)\underset{N\to\infty}{\to}\mathrm{e}^{\langle\log z\rangle} \tag{21}$$

and, therefore,

$$F(0)=\max(0,\langle\log z\rangle). \tag{22}$$

In all our calculations we shall restrict ourselves to the case $\langle\log z\rangle<0$ (equation (3a)), so that

$$F(0)=0. \tag{23}$$

The extension to the case $\langle\log z\rangle>0$ is obvious because one has, in general,

$$F(\varepsilon)=\langle\log z\rangle+G(\varepsilon) \tag{24}$$

where

$$G(\varepsilon)=\frac{1}{N}\log\operatorname{Tr}\prod_{i=1}^{N}\begin{pmatrix}1 & \varepsilon\\ \varepsilon/z_i & 1/z_i\end{pmatrix} \tag{25}$$

and one can always calculate either F or G subject to the restriction (3a). In most examples the case $\langle\log z\rangle=0$ corresponds to a critical point.

3. Breakdown of Taylor expansion of $F(\varepsilon)$

A natural first approach is to assume that $F(\varepsilon)$ is regular near $\varepsilon=0$ and to try to expand $F(\varepsilon)$ in powers of ε^2. Although we shall conclude that such an expansion is not always possible, it is instructive to see why it breaks down. This happens in particular when (3) holds. In most applications the fact that some of the expansion coefficients are infinite has a direct physical meaning.

The simplest way to expand $F(\varepsilon)$ is to assume that the $\log R_i$ in (18) have an expansion in powers of ε^2,

$$\log R_i = A_i\varepsilon^2 + B_i\varepsilon^4 + \ldots . \tag{26}$$

Let $\langle A\rangle, \langle B\rangle, \ldots$ be defined by

$$\langle A\rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} A_i \qquad \langle B\rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} B_i \tag{27a, b}$$

and so on. From equations (21) and (26) we find an expansion of $F(\varepsilon)$:

$$F(\varepsilon) = \langle A\rangle\varepsilon^2 + \langle B\rangle\varepsilon^4 + \ldots . \tag{28}$$

This expansion is meaningful only if $\langle A\rangle, \langle B\rangle, \ldots$ exist. If we substitute the expansion (26) in the recursion (17), we find recursion relations for $A_i, B_i, \ldots$, of which the first ones are

$$A_{i+1} = z_i + z_i A_i \tag{29}$$

$$B_{i+1} + \tfrac{1}{2}(A_{i+1}^2) = -z_i A_i + z_i B_i - \tfrac{1}{2} z_i A_i^2. \tag{30}$$

These relations show that $A_{i+1}, B_{i+1}, \ldots$ are all functions only of the z_k for $k \leqslant i$. Therefore, we have

$$\langle A_i z_i\rangle = \langle A\rangle\langle z\rangle \qquad \langle B_i z_i\rangle = \langle B\rangle\langle z\rangle \tag{31a, b}$$

and so on. From equations (29) and (31a) one has

$$\langle A\rangle = \langle z\rangle(1 + \langle A\rangle) \tag{32}$$

which can be iterated:

$$\langle A\rangle = \langle z\rangle(1 + \langle z\rangle + \langle z\rangle^2 + \ldots) = \langle z\rangle/(1 - \langle z\rangle). \tag{33}$$

The series converges only if

$$\langle z\rangle < 1. \tag{34}$$

This means that $\langle A\rangle$ is finite only if $\langle z\rangle$ is less than one. Similarly one can calculate $\langle B\rangle$ from equations (30) and (31) to find that

$$\langle B\rangle = -\tfrac{1}{2}[(1 + \langle z\rangle)^2\langle z^2\rangle + 2\langle z\rangle^2(1 - \langle z^2\rangle)]/(1 - \langle z\rangle)^2(1 - \langle z^2\rangle). \tag{35}$$

Again $\langle B\rangle$ is finite only if

$$\langle z^2\rangle < 1 \tag{36}$$

and $\langle B\rangle$ is infinite otherwise.

In general one should expect that the coefficient of ε^{2n} is finite only if $\langle z^n\rangle < 1$. Therefore, a full Taylor expansion of $F(\varepsilon)$ is possible only if the moments $\langle z^n\rangle$ are less than one for all integer n. If only a finite number n_0 of moments $\langle z^n\rangle$ are less than one, then one finds only n_0 terms in the expansion. The extreme case on which we concentrate our work is when the first moment $\langle z\rangle$ is larger than one (see (3b)). We investigate this situation in §§ 4 and 5 and conclude that for $\varepsilon \to 0$ one has the power-law behaviour $F(\varepsilon) \sim \varepsilon^{2\alpha}$ with $\alpha < 1$. It is probably true that when there are n_0 moments less than unity, one finds a non-analytic term $\varepsilon^{2\alpha}$ with $n_0 < \alpha < n_0 + 1$. We also expect that as one varies the distribution $\rho(z)$, the coefficient of ε^{2n_0} diverges when $\langle z^{n_0}\rangle \to 1^-$ (see equations (33) and (35)). However, we leave these last two statements as conjectures.

A physical example of the divergence of the quantities $\langle A\rangle, \langle B\rangle, \ldots$ was given by Derrida and Hilhorst (1981), who showed that different powers of the pair correlation of a random chain are singular at different temperatures.

If we consider $F(\varepsilon)$ in a pure case, i.e. $\rho(z)$ is a delta function, then the value $z=1$ is obviously a critical point. In the random case non-analyticity appears as soon as $\rho(z)$ permits values on both sides of the critical point $z=1$ of the pure system. This is very reminiscent of the Griffiths singularities (Griffiths 1969).

4. A simple example

In order to understand the case where condition (3) is fulfilled, it is useful to have an example which can be solved completely. Such an example is furnished by the distribution

$$\rho(z)=(1-p)\delta(z)+p\delta(z-y). \tag{37}$$

Condition (3) then becomes

$$py>1 \qquad p<1. \tag{38}$$

The reason why this example is simpler is that with probability $1-p$, the random matrix is a projector, and the ratio R_{i+1} in (17) is set back to one. For this example one can find explicitly the distribution $P(R)$ satisfying (20), namely

$$P(R)=(1-p)\sum_{n=0}^{\infty} p^n\delta(R-R_n) \tag{39}$$

where the R_n are constructed by the rule

$$\begin{aligned} &R_0=1\\ &R_{n+1}=1+y+y(\varepsilon^2-1)/R_n. \end{aligned} \tag{40}$$

Because the R_n obey a homographic relation, one can write also for $n\geqslant 0$

$$R_n=\frac{1+b}{1+b\lambda}\,\frac{1+b\lambda^{n+1}}{1+b\lambda^n} \tag{41}$$

with

$$b=1+\frac{(1-y)^2}{2\varepsilon^2 y}\left[1-\left(1+4\frac{\varepsilon^2 y}{(1-y)^2}\right)^{1/2}\right] \tag{42}$$

and

$$\lambda=(y-b)/(1-by). \tag{43}$$

From (39), (41) and (19), one finds for $F(\varepsilon)$

$$F(\varepsilon)=p\log(1+b)+p(p-2)\log(1+b\lambda)+(1-p)^2\sum_{n=1}^{\infty}p^n\log(1+b\lambda^{n+1}). \tag{44}$$

The problem is reduced to finding the $\varepsilon\to 0$ behaviour of (44). In that limit we have

$$b\simeq\varepsilon^2 y/(1-y)^2 \qquad \lambda\simeq y. \tag{45}$$

In the sum (44), the first two terms are analytic at $\varepsilon = 0$ and behave like ε^2 for $\varepsilon \to 0$. This means that if there is any singular behaviour, it must come from the series in (44). In appendix 1, we study in detail the $b \to 0$ limit of this series.

We show that because $py > 1$, one has

$$\sum_{n=1}^{\infty} p^n \log(1+by^n) \simeq b^\alpha H\left(\frac{\log b}{\log y}\right) \tag{46}$$

where the exponent α is given by

$$\alpha = -\log p/\log y \tag{47}$$

and H is a periodic function with period 1 given explicitly in appendix 1.

Using the result (46) in (44) and replacing b and λ by their $\varepsilon \to 0$ behaviour (45) one finds

$$F(\varepsilon) \simeq \varepsilon^{2\alpha} H\left(\frac{2\log\varepsilon}{\log y}\right). \tag{48}$$

Clearly, the relations (47) and (37) can be written in the form of equation (5). We shall see that the power law $\varepsilon^{2\alpha}$ found here appears also for general distributions $\rho(z)$ and that α is always given by (5). On the contrary the periodic function H seems very special to this example and can occur only for particular distributions. This special behaviour has also been found recently by Bernasconi and Schneider (1982) in a hopping problem for a particular choice of the distribution of hopping rates.

5. The general case

We shall now investigate the behaviour of $F(\varepsilon)$ in the limit $\varepsilon \to 0$ when the distribution $\rho(z)$ is any distribution which satisfies condition (3). To this end, we first study the behaviour of the distribution $P(R)$ as $\varepsilon \to 0$ and then calculate the singularity of $F(\varepsilon)$ using formula (19). It is useful to put

$$R = 1 + r \tag{49}$$

$$Q(r) = P(1+r) \tag{50}$$

and to consider the integral equation satisfied by the distribution $Q(r)$ of r. From (20), one finds

$$Q(r) = \int \rho(z)\,\mathrm{d}z \frac{(1-\varepsilon^2)z}{(z-r)^2} Q\left(\frac{r - z\varepsilon^2}{z-r}\right). \tag{51}$$

It may be that for some particular $\rho(z)$ like the one in § 4, one can solve the integral equation (51) explicitly. However, there is no hope of solving (51) exactly for arbitrary distribution $\rho(z)$.

Our approach to the general case is based on making a distinction between two different regions of r: in region I, the values of r are of order ε^2 whereas in region II, r is of order unity. We observe that whenever r is in region I (region II), then the argument of Q on the right-hand side of equation (51) is also in region I (region II). Thus the integral equation (51) effectively couples each region only to itself, and we may consider it in each region separately. This way of reasoning tacitly assumes that z is always of order one, which will be the case for functions $\rho(z)$ that vanish sufficiently rapidly as $z \to 0$. Other cases (like the one of the preceding section) should, in principle, be examined separately.

In region I we put

$$r = \varepsilon^2 s \tag{52}$$

with s finite. If we use (52), expand (51) as a power series in ε^2, and keep only the lowest-order terms, we find

$$Q(\varepsilon^2 s) = \int \mathrm{d}z\, \rho(z) \frac{1}{z} Q\left(\frac{\varepsilon^2(s-z)}{z}\right). \tag{53}$$

If we put

$$Q(\varepsilon^2 s) = g_1(s), \tag{54}$$

then the function g_1 satisfies an equation independent of ε,

$$g_1(s) = \int \rho(z)\, \mathrm{d}z \frac{1}{z} g_1\left(\frac{s-1}{z}\right). \tag{55}$$

In region II, i.e. r of order unity, one can expand equation (51) in a similar way to find that $Q(r)$ can be written as

$$Q(r) = g_2(r) \tag{56}$$

where g_2 also satisfies an equation independent of ε,

$$g_2(s) = \int \rho(z)\, \mathrm{d}z \frac{z}{(z-s)^2} g_2\left(\frac{s}{z-s}\right). \tag{57}$$

Even if equations (55) and (57) are simpler than equation (51), they are still too complicated to be solved for a general distribution $\rho(z)$. However, we shall see that knowledge of the asymptotic behaviour of the solutions is sufficient to obtain $F(\varepsilon)$ when $\varepsilon \to 0$.

Since the two functions g_1 and g_2 describe the same distribution $P(R)$, the asymptotic behaviour of $g_1(r/\varepsilon^2)$ for $r \to \infty$ should be identical to the asymptotic behaviour of $g_2(r)$ for $r \to 0$.

For $s \to \infty$, the function g_1 satisfies asymptotically

$$g(s) = \int \rho(z) z^{-1}\, \mathrm{d}z\, g(s/z). \tag{58}$$

This is just equation (55) with $s-1$ replaced by s. Similarly, in the limit $r \to 0$, g_2 also satisfies (58) but with a different definition of s. Equation (58) is a convolution in the variable $\log s$. Therefore, the general solution of (58) is

$$g(x) = \sum_i C_i x^{-\alpha_i - 1} \tag{59}$$

where the C_i are arbitrary constants and the sum runs over all the solutions α_i of the equation

$$\int \rho(z) z^{\alpha}\, \mathrm{d}z = 1. \tag{60}$$

It is easy to check that the left-hand member of (60) is a convex function of α. Therefore, there are, in general, at most two real solutions of (60): the trivial solution

$\alpha = 0$ and possibly another real one $\alpha = \alpha^*$. Because of (3) we know that α^* exists and is positive; it will be the interesting solution. In addition to these two real solutions, there are in general many complex ones. Some examples where all the α_i can be found explicitly are discussed in appendix 2.

We shall assume that g_1 and g_2 have the asymptotic behaviour

$$g_1(s) \simeq A_1 s^{-\alpha^*-1} \qquad \text{for} \qquad s \to \infty \tag{61a}$$

$$g_2(s) \simeq A_2 s^{-\alpha^*-1} \qquad \text{for} \qquad s \to 0 \tag{61b}$$

where A_1 and A_2 are constants. We impose here the special condition

$$\rho(z) \neq \sum_n p_n \delta(z - \lambda^n) \tag{62}$$

which excludes certain distributions $\rho(z)$ discussed in appendix 2. In order to justify (61), we note that the complex α_i have their real part, when it is positive, larger than α^* (see appendix 2). However, there are complex roots with negative real parts and there is also the root $\alpha = 0$. It is not obvious that the constants C_i in (59) are zero for these cumbersome roots and that the constant C^* for the root α^* is not zero. Of course, we can argue that we do not want the leading behaviour to be due to a complex α_i because the functions g_1 or g_2 have to remain positive. However, it is hard to eliminate *a priori* the root $\alpha = 0$. The best justification of (61) is to use a result of Kesten *et al* (1975), who studied the distribution of a variable x_i given by

$$x_i = \sum_{n=1}^{\infty} \prod_{j=1}^{n} z_{i-j} \tag{63}$$

where the z_i are independent random positive numbers distributed according to a given distribution $\rho(z)$ which satisfies condition (62). These authors proved that the distribution of x_i falls off like $x_i^{-\alpha^*}$ for large x_i. It is easy to check that if the difference between R_i and 1 is of order ε^2, and one puts

$$R_i = 1 + x_i \varepsilon^2, \tag{64}$$

then the variable x_i is precisely given by (63) when one substitutes (64) in (17).

Once the behaviours (61) have been accepted, the calculation of $F(\varepsilon)$ for $\varepsilon \to 0$ becomes easy. The matching condition implies that

$$A_1 \left(\frac{r}{\varepsilon^2}\right)^{-\alpha^*-1} \simeq A_2 r^{-\alpha^*-1}; \tag{65}$$

therefore,

$$A_1 = \varepsilon^{-2\alpha^*-2} A_2. \tag{66}$$

Let us define the solutions G_1 and G_2 of equations (55) and (57) by

$$G_1(s) = g_1(s)/A_1 \qquad G_2(s) = g_2(s)/A_2 \tag{67}$$

where A_1 and A_2 have been defined in (61). This means that we normalise G_1 and G_2 in such a way that

$$\begin{aligned} G_1(s) &\simeq s^{-1-\alpha^*} \qquad \text{for} \qquad s \to \infty \\ G_2(s) &\simeq s^{-1-\alpha^*} \qquad \text{for} \qquad s \to 0. \end{aligned} \tag{68}$$

Of course G_1 and G_2 do not depend on ε. The function $P(1+r)$ that we seek to calculate is then given by

$$P(1+r)=\begin{cases} A_2\varepsilon^{-2\alpha^*-2}G_1(r/\varepsilon^2) & \text{region I} \\ A_2G_2(r) & \text{region II.} \end{cases} \tag{69}$$

In order to determine the normalisation constant A_2 we choose an arbitrary Λ which separates regions I and II. This Λ has to satisfy the condition

$$\varepsilon^2 \ll \Lambda \ll 1. \tag{70}$$

The normalisation becomes

$$A_2\left(\varepsilon^{-2\alpha^*-2}\int_0^\Lambda G_1(r/\varepsilon^2)\,\mathrm{d}r+\int_\Lambda^\infty G_2(r)\,\mathrm{d}r\right)=1. \tag{71}$$

Of course the dependence on Λ of the first integral is cancelled by the second integral because of (61), (66) and (70). In the limit $\varepsilon\to 0$, one finds that A_2 is given by

$$A_2 \simeq \varepsilon^{2\alpha^*}\left(\int_0^\infty G_1(r)\,\mathrm{d}r\right)^{-1}. \tag{72}$$

We can now calculate $F(\varepsilon)$ from (19) and (50). One has

$$F(\varepsilon)\simeq A_2\left(\varepsilon^{-2\alpha^*-2}\int_0^\Lambda g_1(r/\varepsilon^2)\log(1+r)\,\mathrm{d}r+\int_\Lambda^\infty g_2(r)\log(1+r)\,\mathrm{d}r\right). \tag{73}$$

Here again the Λ dependence disappears and one obtains

$$F(\varepsilon)\simeq A_2\int_0^\infty \log(1+r)G_2(r)\,\mathrm{d}r. \tag{74}$$

By combining equations (72) and (74) we arrive at our final expression for $F(\varepsilon)$:

$$F(\varepsilon)\simeq\varepsilon^{2\alpha^*}\left(\int_0^\infty \log(1+r)G_2(r)\,\mathrm{d}r\Big/\int_0^\infty G_1(r)\,\mathrm{d}r\right) \qquad \varepsilon\to 0 \tag{75}$$

where $0<\alpha^*<1$ (for $\alpha^*>1$ analytic contributions $\simeq\varepsilon^2$ dominate). Thus $F(\varepsilon)$ exhibits the non-analytic behaviour announced in the introduction, with an exponent α^* that depends continuously on the distribution $\rho(z)$. The proportionality constant in (75) is fully defined, but its explicit calculation involves the complete solution of the integral equations (55) and (57); we were able to solve these only in the asymptotic form (58).

Let us now come back to the distributions $\rho(z)$ of the form

$$\rho(z)=\sum p_n\delta(z-\lambda^n). \tag{76}$$

Since in this case there are complex α_i with real part equal to α^* (see appendix 2), we expect g_1 to behave like

$$g_1(s)\simeq s^{-\alpha^*-1}H(\log s/\log\lambda) \qquad \text{for} \qquad s\to 0 \tag{77}$$

where H is a periodic function of period one. We shall not discuss this case because the matching of g_1 and g_2 is more complicated. However, it is probably true that, like in the example of § 4, the power law found in (75) remains the same but the constant is replaced by a periodic function of $\log \varepsilon$, like in the hopping problem studied by Bernasconi and Schneider (1982).

6. Applications

We return to the Ising chain with coupling constant J in a randomly site-dependent field h_i, as described by equations (6)–(9). By transposing the results (28) and (33) to Ising notation, we have the large-J expansion

$$\log Z = J + \langle h_i \rangle + \frac{\langle e^{-2h_i} \rangle}{1 - \langle e^{-2h_i} \rangle} e^{-4J} + \ldots \tag{78}$$

whenever

$$\langle e^{-2h_i} \rangle < 1 \qquad \text{(case I)}. \tag{79}$$

From (75) we have the complementary result

$$\log Z = J + \langle h_i \rangle + C\, e^{-4\alpha J} + \ldots \tag{80}$$

whenever

$$\langle e^{-2h_i} \rangle > 1 \qquad \text{and} \qquad \langle h_i \rangle > 0 \qquad \text{(case II)}. \tag{81}$$

Here C is a constant depending on the probability distribution $\tilde{\rho}(h)$ of the fields, and α is the solution of

$$\int \mathrm{d}h\, e^{-2\alpha h} \tilde{\rho}(h) = 1. \tag{82}$$

Equation (81) implies that in case II not all the h_i have the same sign, and, therefore, that a spin configuration cannot satisfy all couplings and all magnetic fields simultaneously. The non-analytic behaviour (80) is thus seen to be directly connected to a frustration effect. By differentiating $\log Z$ in (78) and (80) with respect to the fields and to the coupling constant, one finds expressions for the average magnetisation and internal energy per site. These determine directly the fraction of negative spins and the density of sequences of negative spins. In both cases, I and II, there is a net positive magnetisation, tending towards one as $J \to \infty$, and the density of overturned spin sequences vanishes in that limit. However, in case I the average length of such sequences tends toward a constant, whereas in case II the sequences become infinitely long as $J \to \infty$.

A second possible application is to the random hopping problems mentioned in the introduction. For these, the non-analytic behaviour (75) manifests itself in the long time dependence of the displacement (Kesten *et al* 1975, Derrida and Pomeau 1982, Bernasconi and Schneider 1982) and should also be seen in the frequency dependence of the response to an external force. We do not work this out in detail.

Finally, the results of this paper can be used to calculate the boundary magnetisation and correlation functions of a randomly layered two-dimensional Ising model of the McCoy and Wu type (see § 1) for an arbitrary distribution of the random couplings. The result is that one finds a non-analytic field dependence of the magnetisation, and a divergent boundary susceptibility in a *finite* region around the critical point. We hope to present details on this application in a future note.

Acknowledgments

One of us (BD) would like to thank J Avron, M A Moore and L Thomas for discussions on this problem at the Aspen Center of Physics.

Appendix 1

We study the limiting behaviour of the function

$$f(b) = \sum_{n=1}^{\infty} p^n \log(1 + y^n b) \tag{A1.1}$$

in the limit $b \to 0$. This function appears in the expression (44) for $F(\varepsilon)$ and we are in the case where

$$p < 1 \qquad \text{and} \qquad py > 1. \tag{A1.2}$$

To estimate the sum (A1.1) the simplest way is to use the saddle-point method and to find the values of n which dominate (A1.1). By differentiating $p^n \log(1+by^n)$ with respect to n, one finds a value $\tilde{n}$ for which this expression has a maximum:

$$(\log p) \log(1 + y^{\tilde{n}} b) + b y^{\tilde{n}} \frac{\log y}{1 + y^{\tilde{n}} b} = 0. \tag{A1.3}$$

Let u be the positive solution of

$$\log p \log(1+u) + u(1+u)^{-1} \log y = 0. \tag{A1.4}$$

This solution exists and is unique because of condition (A1.2). Then $\tilde{n}$ is given by

$$\tilde{n} = (\log u - \log b)/\log y. \tag{A1.5}$$

Of course, $\tilde{n}$ has no reason to be an integer. Therefore, the largest term will be one of the two integers surrounding $\tilde{n}$. Anyhow, the sum (A1.1) is well approximated by the sum of the terms corresponding to all the integers at a finite distance of $\tilde{n}$. It is useful to decompose $\tilde{n}$ into

$$\tilde{n} = n_0 + x \tag{A1.6}$$

where n_0 is an integer and x a real number such that

$$0 \leqslant x < 1. \tag{A1.7}$$

We can write $f(b)$ without approximation as

$$f(b) = \sum_{m=-n_0+1}^{\infty} p^{n_0+m} \log(1 + y^{n_0+m} b). \tag{A1.8}$$

Let us replace n_0 in (A1.8) by $\tilde{n} - x$ and use (A1.5). We then get

$$f(b) = \left(\frac{b}{u}\right)^{-(\log p/\log y)} \sum_{m=-n_0+1}^{\infty} p^{m-x} \log(1 + u y^{m-x}). \tag{A1.9}$$

The expression (A1.9) is exact. Since in (A1.5) the parameters u and y are fixed, n_0 becomes very large for $b \to 0$ and we can replace, in that limit, the lower value of m

in the sum (A1.9) by $-\infty$. Therefore,

$$f(b) \simeq b^{-(\log p/\log y)} H(x) \tag{A1.10}$$

where $H(x)$ is given by

$$H(x) = u^{(\log p/\log y)} \sum_{m=-\infty}^{+\infty} p^{m-x} \log(1+uy^{m-x}). \tag{A1.11}$$

Clearly $H(x)$ is a periodic function of period one:

$$H(x) = H(x+1). \tag{A1.12}$$

Therefore, because of (A1.5) and (A1.6) it is a periodic function of $\log b/\log y$ with the same period.

Appendix 2

We study the complex solutions α of the equation

$$\int z^\alpha \rho(z)\, \mathrm{d}z = 1 \tag{A2.1}$$

for several distributions $\rho(z)$. Under the condition (3), which we assume, the real solutions are limited to $\alpha = 0$ and a non-trivial positive solution $\alpha = \alpha^*$. Let us start with the distribution studied in § 4,

$$\rho(z) = (1-p)\delta(z) + p\delta(z-y). \tag{A2.2}$$

Equation (A2.1) then becomes

$$py^\alpha = 1. \tag{A2.3}$$

The solutions α are given by

$$\alpha = -(\log p/\log y) + (2\mathrm{i}\pi/\log y)n \qquad n \in Z. \tag{A2.4}$$

So in this case, all the α have the same real part.

If $\alpha = \alpha_1 + \mathrm{i}\alpha_2$ is a root of (A2.1) for a general distribution $\rho(z)$, one must have

$$\int \rho(z)\, \mathrm{d}z\, z^{\alpha_1} (\cos(\alpha_2 \log z) + \mathrm{i} \sin(\alpha_2 \log z)) = 1. \tag{A2.5}$$

To have a complex α with real part equal to α^*, $\cos(\alpha_2 \log z)$ must be equal to one on the whole support of the distribution $\rho(z)$. The only distributions for which this happens are of the form

$$\rho(z) = \sum_{n=-\infty}^{+\infty} p_n \delta(z - \lambda^n) \tag{A2.6}$$

with

$$\sum_{n=-\infty}^{+\infty} p_n = 1 \tag{A2.7}$$

and λ arbitrary. The example (A2.2) belongs to this family with $\lambda = y$, $p_1 = p$ and $p_{-\infty} = 1-p$.

In general, if the distribution $\rho(z)$ is not of the form (A2.6), then one sees from (A2.5) that the complex roots α of (A2.1) must have their real parts larger than the real positive solution, or less than zero. To illustrate that, let us give an example for which all the α_i can be found explicitly. Let

$$\rho(z) = \exp[-\tfrac{1}{2}a(\log z + b/a)^2] \Big/ \int_0^\infty \mathrm{d}z' \exp[-\tfrac{1}{2}a(\log z' + b/a)^2] \tag{A2.8}$$

where $a > 0$ and $b > 1$ are arbitrary constants. Upon using (A2.8) in (A2.1) and putting $y = \log z$, we obtain for α the equation

$$\int_{-\infty}^{\infty} \mathrm{d}y \exp[-\tfrac{1}{2}ay^2 + (-b+1+\alpha)y] = \int_{-\infty}^{\infty} \mathrm{d}y \exp[-\tfrac{1}{2}ay^2 + (-b+1)y]. \tag{A2.9}$$

Both integrals are Gaussian and on carrying them out we find an equation for α which can be solved to give the infinity of solutions

$$\alpha_n^\pm = b - 1 \pm [(b-1)^2 + 4\pi n a \mathrm{i}]^{1/2} \qquad n = 0, \pm 1, \pm 2, \ldots . \tag{A2.10}$$

These lie on the two branches of the hyperbola given by

$$\left(\frac{\mathrm{Re}\,\alpha}{b-1} - 1\right)^2 - \left(\frac{\mathrm{Im}\,\alpha}{b-1}\right)^2 = 1. \tag{A2.11}$$

The real roots are $\alpha = 0$ and $\alpha^* = 2(b-1)$, and the complex roots have their real parts either negative or larger than α^*.

References

Alexander S, Bernasconi J, Schneider W R and Orbach R 1981 *Rev. Mod. Phys.* **53** 175
Bernasconi J and Schneider W R 1982 *J. Phys. A: Math. Gen.* **15** L729
Derrida B 1983 *J. Stat. Phys.* **31** in press (June)
Derrida B and Hilhorst H J 1981 *J. Phys. C: Solid State Phys.* **14** L539
Derrida B and Pomeau Y 1982 *Phys. Rev. Lett.* **48** 627
Dyson F J 1953 *Phys. Rev.* **92** 1331
Furstenberg H 1963 *Trans. Am. Math. Soc.* **108** 377
Griffiths R B 1969 *Phys. Rev. Lett.* **23** 17
Kesten H, Kozlov M V and Spitzer F 1975 *Compositio Mathematica* **30** 145
McCoy B M 1969 *Phys. Rev.* **188** 1014
—— 1972 *Phase Transitions and Critical Phenomena* ed C Domb and M S Green vol 2, ch 5 (London: Academic)
McCoy B M and Wu T T 1969 *Phys. Rev.* **188** 982
—— 1973 *The Two-Dimensional Ising Model* (Cambridge, Mass.: Harvard University Press) ch XIV and XV
Ruelle D 1979 *Adv. Math.* **32** 68
Ziman T A L 1982 *Phys. Rev. Lett.* **49** 337